

Gridding for Non-Cartesian k-space Sampling

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Nearly all routine clinical imaging in MRI is based on Cartesian sampling (measuring Fourier data on a regularly spaced rectilinear grid). This is true despite a large body of work has illustrated the potential benefits of various non-Cartesian methods (radial, spiral, etc.) in a variety of applications. These can include scan efficiency (maximum k-space coverage per time), robustness against motion, the ability to minimize T2^(*) decay in the center of k-space, and general flexibility in measurement strategies. There are various reasons for the nearly complete lack of non-Cartesian methods in routine practice, one of them being the fidelity and simplicity of reconstruction (the subject of this abstract).

Reconstruction of Cartesian-sampled data is easy. If data are collected on a regularly spaced rectilinear grid in k-space, then (1) reconstruction can be performed rapidly using the Fast Fourier transform, (2) k-space is measured uniformly, which prevents noise coloring and amplification in the image, (3) the Nyquist sampling criterion is well understood and can be met exactly, and (4) the point spread function (PSF) is well-behaved, and resolution is easily determined. All of this results in rapid reconstruction of an 'exact' image consistent with the measured data which is well characterized, and minimizes noise effects.

There has been a great deal of work in developing reconstruction methods for data from Non-Cartesian methods, to increase speed and fidelity, as well as to characterize noise behavior and resulting PSF's (which reflect both resolution and aliasing).

2. Gridding Basics

The image $f(x,y)$ is defined as the Fourier transform of it's Fourier representation $F(k_x,k_y)$ by the Fourier equation

$$f(x, y) = \int F(k_x, k_y) \exp^{i2\pi[k_x x + k_y y]} . \quad [1]$$

For gridding reconstruction, this is typically estimated using four steps^[1]: sampling density compensation, interpolating the data onto a Cartesian grid (the "gridding" process), Fourier transforming the data, and applying a rolloff correction filter. As outlined in [1], density compensation and gridding may be mathematically expressed as:

$$F_g = [(FSW) \otimes C]R, \quad [2]$$

where F_g is the data after gridding, S is a series of delta functions in k-space to represent the sampling locations, W is a sampling density weighting correction applied to the delta functions, C is the convolution kernel used for gridding, R is a rectangular grid of delta functions used for re-sampling, and \otimes is the convolution operator. This is very well explained in reference [1], and illustrated in Fig. 1.

2.1. Gridding

This process takes place after Sampling Density Compensation, but it is easier to consider this process first. To make use of the FFT, measured data F are interpolated

Fig. 1. Effects of sampling and gridding reconstruction.

Column A:
 $F(k_x, k_y)$
 (magnitude raised to 0.25 power for visualization).

Column B: $f(x, y)$
 (image FOV = 64 pixels, image zero-padded to 512 pixels for illustration)

Column C: 3D rendering of $f(x, y)$.

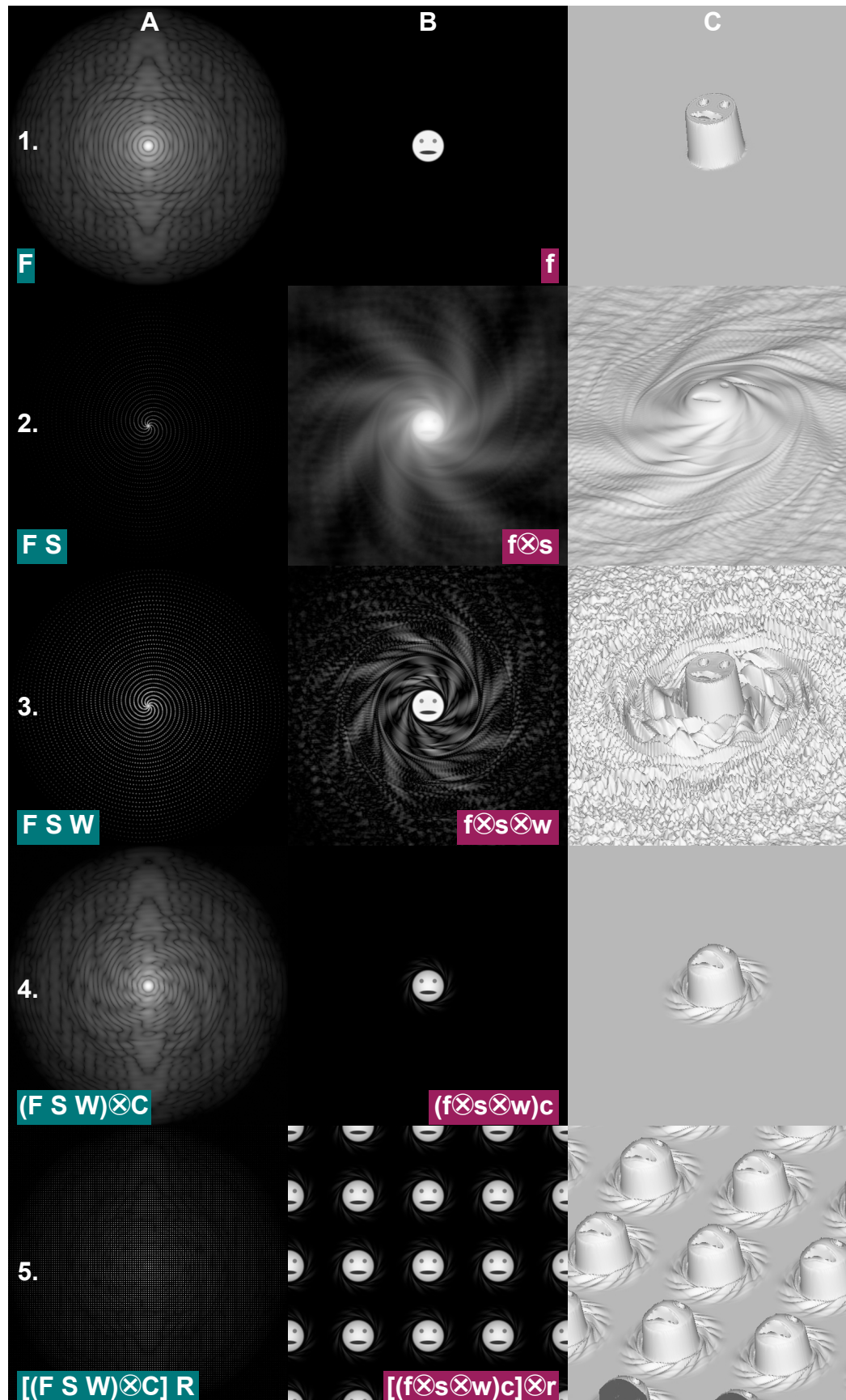
Row 1: Original data.

Row 2: Measured (sampled) data.

Row 3: Row 2 data after density correction.

Row 4: Row 3 data after convolution with C.

Row 5: Row 4 data after sampling on Cartesian grid.



onto a Cartesian Grid using a convolution kernel c , so that the interpolated data F_g are given by a weighted summation of all neighboring measure data, i.e.

$$F_g(i, j) = \sum F(n) c[k_x(n), k_y(n), k_x(i, j), k_y(i, j)], \quad [3]$$

where c generally weights the contributions of $F(n)$ (the n th measured data point) to F_g based on the distance from the measured sample point $\{k_x(n), k_y(n)\}$ to the point on the Cartesian grid $\{k_x(i, j), k_y(i, j)\}$. Note that Eq. [2] represents the convolution by C and the multiplication by R in Eq. [2]. The convolution kernel can be fixed^[1], or it vary in k -space based on some measure of optimality^[2,3].

The choice of C and R trades off reconstruction efficiency with image accuracy (a wider C and more densely packed R reduce error but increase gridding time and post-gridding FFT time, respectively). An excellent review of this is given in reference [4].

2.2 Sampling Density Compensation:

The gridding process, as it is written in Eq. [2], will generally make the value of F_g proportional to the sampling density. For this reason, data are normalized by the inverse of their sampling density prior to gridding. Although this can in principle be done after gridding, it is generally necessary to do this first to retain accuracy in the gridding process. Figure 2 gives a simple illustration of the benefits of sampling density prior to gridding. The value of F_x when gridded prior to sampling density compensation (Fig. 1a) will be skewed toward that of F_1 and F_2 , while the value of F_x when gridded after to sampling density compensation (Fig. 1b) will be more properly estimated.

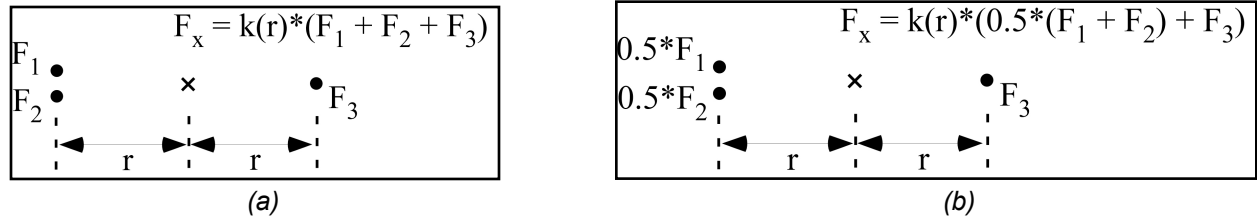


Fig. 2. Specific example of increased gridding accuracy after density compensation. Data are sampled at locations solid dots 1, 2, and 3 (corresponding to values F_1 , F_2 , and F_3 , respectively). Locations 1 and 2 are close together, so that $F_1 = F_2$ (plus noise); all three sampling locations are equidistant (distance = r) from the point to be estimated, denoted by “ x ”. Without sampling density, (a) the estimate F_x is proportional to $2 F_1 + F_3$. After sampling density, (b) the values of F_1 and F_2 are reduced by 2, and the estimate is now correctly proportional to the average of $F_1 + F_3$.

There are a variety of methods for calculating the sampling density, including analytic estimates, geometric estimates such as the Voronoi method^[5] and numerical methods such as proposed by Jackson^[1] and later extended by Pipe^[6].

2.3 Rolloff Filter:

The gridding convolution by C effectively multiplies the image by its Fourier pair $c(x, y)$, which tends to reduce the image values around the edges (see Fig. 1, row 4, column C). The final step to gridding reconstruction is division of the image by the rolloff filter - this “rolloff correction” is shown in Fig. 3.

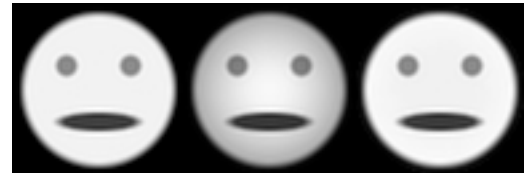


Fig. 3. Original image (left), after gridding and FFT (middle), and after rolloff correction (right).

3. ADDITIONAL CONSIDERATIONS

3.1. *What is the Nyquist Criterion?*

It is helpful to appreciate the complexities of reconstruction by considering the Nyquist criterion in 2D non-Cartesian imaging. The Nyquist sampling limit applies to regularly spaced sampling intervals - when data are sampled in a non-regular fashion, this limit no longer strictly applies. One can consider that, given a supported FOV in image space, each sampled point is strongly correlated with the values of F within roughly $1/FOV$ of that point in k -space, and weakly correlated (to some degree) with points beyond the local neighborhood. Linear reconstruction methods are generally approximate attempts at matrix inversion, and areas in k -space that are not strongly correlated with any sampling points can cause singularities in the inverted "reconstruction" matrix (and thereby can create significant noise amplification) unless appropriate regularization is used (e.g. do not use these areas of k -space). The equivalent of the Nyquist Criterion, in qualitative terms, is that sampled points are put in a fashion that makes the matrix inversion the most stable.

3.2. *Undersampled and Oversampled Data.*

When data are undersampled (e.g. do not meet this loose Nyquist Criterion) in parts of k -space, one must consider the effects of data weighting and reconstruction. Data weighting can emphasize existing sampling points within the undersampled regions, which can improve resolution (i.e. improve the shape of the inner part of the PSF) at the expense of increased aliasing (i.e. increase the outer signal in the PSF, which creates aliasing).^[6]

For oversampled data, data weighting can be adjusted so that "redundant" samples are weighted evenly, to maximize SNR, or data can be preferentially weighted based on some criterion (e.g. data "goodness", temporal weightings, etc.)

3.3. *Assessing Non-Cartesian reconstruction.*

There are several criterion that would benefit the MRI community in assessing Non-Cartesian reconstruction, such as metrics for resolution between differently shaped PSF's, metrics for colored noise/SNR, general metrics for aliasing error, and a quantitative description of either the Nyquist Criterion or of reconstruction ("inversion") stability.

4. REFERENCES

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